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Projector equivalences in K theory and families of non-commutative solitons

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Abstract: Projector equivalences used in the definition of the K-theory of operator algebras are shown to lead to generalizations of the solution generating technique for solitons in NC field theories, which has recently been used in the construction of branes from other branes in B-field backgrounds and in the construction of fluxon solutions of gauge theories. The generalizations involve families of static solutions as well as solutions which depend on euclidean time and interpolate between different configurations. We investigate the physics of these generalizations in the brane-construction as well as the fluxon context. These results can be interpreted in the light of recent discussions on the topology of the configuration space of string fields.

Keywords: Solitons Monopoles and Instantons, D-branes, Non-Commutative Geometry.

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Contents

1. Introduction

Solutions to non-commutative field theories on brane worldvolumes have been of great interest recently [1]–[9]. Charges of branes and the spacetime fields they couple to have been shown to be closely related to K-theory [10]–[12]. In this paper we explore applications of some standard techniques in the K-theory of operator algebras in the context of non-commutative solitons. While the technical relations between the two that we develop are straightforward and of physical interest, a bigger picture which would explain these relations is not entirely clear.

A useful method for generating new solutions to noncommutative field theories (NCFT) from old ones has recently been exploited in [13] to construct lower dimensional brane solutions from known vacuum solutions in string theory. The technique however is quite general and has been applied in many other contexts as well, see e.g. [13]–[16]. The idea is most easily formulated in the matrix or Hilbert space representation of NCFT's discussed in [3]. Conjugation of the fields in the theory by some operator in the noncommutative Hilbert space U as in e.g.

$$\phi \to U\phi U^\dagger \tag{1.1}$$

implies that the action S transforms as

$$S = \int \mathrm{Tr}\mathcal{L} \to S_U = \int \mathrm{Tr}(U\mathcal{L}U^\dagger) \tag{1.2}$$

assuming that $U^\dagger U = 1$. The trace appearing in the action is over the noncommutative Hilbert space as well as any gauge indices. If U can be cycled through the trace then this transformation is a symmetry of the action. However this need not be so, in which case the transformation is not a symmetry of the action but nevertheless remains a symmetry of the equations of motion, i.e.,

$$\frac{\delta S}{\delta\phi} = 0 \to U\frac{\delta S}{\delta\phi}U^\dagger = 0\,. \tag{1.3}$$

Therefore from one solution follows another by conjugation. Actually this holds whether U generates a symmetry of the action or just of the equations of motion. In the former case however U would correspond to a (local or global) gauge transformation so that the "new" solution would either be equivalent to the old one or simply related by a global symmetry transformation.

When U cannot be cycled through the trace it is clear that the old and new solutions are not equivalent in general because their energies differ. This follows simply by noting that the energy functional transforms in the same way as the action functional above and therefore is not invariant.

An important point about the solution generating technique is that it holds regardless of the value of $UU^\dagger$, i.e. U need not be unitary. In fact all the solutions generated by this technique [13]–[15] have used U's which are not unitary but rather which satisfy $UU^\dagger = P$ where P is a projection operator. In the K-theory of operator algebras, projection operators P and Q which are related by

$$U^\dagger U = P,\ UU^\dagger = Q \tag{1.4}$$

are known as von Neumann or algebraically equivalent. It is a straightforward exercise, that we review in section 2, to show that these operators become unitarily equivalent when embedded as projectors in $\mathcal{M}_2(\mathcal{A})$, the algebra of 2×2 matrices with entries in the algebra $\mathcal{A}$. Further, embedding in $\mathcal{M}_4(A)$, we can get a one-parameter family of unitary operators $W(t)$ such that conjugating the embedded operator P leads to a family of projectors which interpolate from the embedded P to the embedded Q. We review these facts in the appendix A and comment on their significance in the K theory of operator algebras.

In physical applications, the embedding into $\mathcal{M}_4(\mathcal{A})$ becomes relevant when we go from a system with U(1) gauge symmetry to one with U(4) gauge symmetry. This involves generalizing a system with a single brane to one involving four branes. The family of interpolating projectors can be used to construct a family of static solutions which interpolate between the trivial solution and the new solution. By mapping the interpolating parameter to the time variable in euclidean space, we can also construct time dependent solutions to the euclidean equations of motion. We consider two classes of such solutions. The first interpolates the solution based on P to the solution based on Q. The second interpolates from the solution based on P back to that based on P after passing through the solution Q.

To get some insight into the physical significance of these solution generating techniques we consider two classes of examples. The first involves solitons in unstable brane systems which are used to construct lower dimensional branes [17, 18, 13, 16] The second involves fluxons, which are limits of non-commutative monopole solutions [19].

2. Background on non-commutative field theories

We begin by reviewing our conventions. The noncommutative coordinates satisfy the commutator conditions

$$[\hat{x}^i, \hat{x}^j] = i\theta^{ij} \,. \tag{2.1}$$

θ^{ij} can be block diagonalized into 2×2 matrices. Suppose this has been done and consider the $i, j = 1, 2$ block and define $\theta^{12} = \theta$. Define the complex noncommutative coordinates

$$z := \frac{\hat{x}^1 + i\hat{x}^2}{\sqrt{2}} \,, \qquad \bar{z} := \frac{\hat{x}^1 - i\hat{x}^2}{\sqrt{2}} \,. \tag{2.2}$$

These coordinates satisfy the commutation relation

$$[z, \bar{z}] = \theta \tag{2.3}$$

so that the annihilation and creation operators defined as

$$a := \frac{z}{\sqrt{\theta}} \,, \qquad a^\dagger := \frac{\bar{z}}{\sqrt{\theta}} \tag{2.4}$$

satisfy the usual commutation relation $[a, a^\dagger] = 1$. The algebra of annihilation and creation operators can be realized on a Hilbert space in the standard way, i.e. let

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} C|n\rangle \tag{2.5}$$

where the action of a and $a^\dagger$ on the $|n\rangle$ state is given by

$$a|n\rangle = \sqrt{n}|n-1\rangle, a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \,. \tag{2.6}$$

Another representation of the algebra is constructed in terms of c-number functions of the x^i's by deforming the multiplication operation to the star product $\star$ defined as

$$f \star g(x) := \exp\left[\frac{i}{2}\theta^{ij}\frac{\partial}{\partial x_1^i}\frac{\partial}{\partial x_2^j}\right] f(x_1)g(x_2)|_{x_1=x_2=x}\,. \tag{2.7}$$

One can map this representation into the above Fock space representation by the Weyl ordering map which takes a function of the c-number coordinates x^i into the associated operator via

$$\hat{f}(a,a^\dagger) = \int \frac{d^2p}{2\pi} e^{i(\bar{p}a + pa^\dagger)}\tilde{f}(p)\,, \tag{2.8}$$

where

$$\tilde{f}(p) = \int \frac{d^2z}{2\pi} e^{-i(\bar{p}z + p\bar{z})} f(z,\bar{z})\,. \tag{2.9}$$

One property of this map is that

$${}_{\mathcal{H}}\hat{f}(a,a^\dagger) = \frac{1}{2\pi\theta}\int d^2z f(z,\bar{z})\,. \tag{2.10}$$

To construct field theories over noncommutative spaces we need derivative operators. Derivatives generate infinitesimal translations of the coordinates

$$\delta_\epsilon x^i = \epsilon^j\partial_j x^i = \epsilon^i\,. \tag{2.11}$$

For noncommutative coordinates however this same action is generated by commuting $\hat{x}^i$ with $-i\epsilon^j\theta_{jk}\hat{x}^k$. Derivatives with respect to noncommutative coordinates can therefore be written as

$$\hat{\partial}_i f(\hat{x}) = [-i\theta_{ij}\hat{x}^j, f(\hat{x})]\,. \tag{2.12}$$

Derivatives with respect to z and $\bar{z}$ are then transformed into commutators with $a^\dagger$ and a respectively via the relations

$$\frac{\partial}{\partial z} = -\frac{1}{\sqrt{\theta}}[a^\dagger,\cdot]\,, \qquad \frac{\partial}{\partial\bar{z}} = \frac{1}{\sqrt{\theta}}[a,\cdot]\,. \tag{2.13}$$

3. Solution generating operators, unitary equivalence and homotopy

Exact solutions are constructed by conjugating a known solution by an operator U satisfying

$$U^\dagger U = 1\,, \qquad UU^\dagger = P \tag{3.1}$$

where P is a projection operator. A particularly simple example is given by the shift operator $S = \sum_{i=0}^{\infty} |i+1\rangle\langle i|$ which satisfies the condition

$$S^\dagger S = 1SS^\dagger = 1 - P_0 \tag{3.2}$$

where $P_0 = |0\rangle\langle 0|$. Acting on operators as $\mathcal{O} \to S\mathcal{O}S^\dagger$ is a symmetry of the equations of motion and allows us to generate new solutions from old ones. It is however not a symmetry of the action because S cannot be cycled through the trace as discussed in the introduction. In general the condition for being able to cycle operators is $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ if A is trace-class and B is bounded. For the shift operator these conditions are not satisfied and we get a different action, and futhermore a different energy.

The equations (3.2) can be expressed as establishing that the identity operator and $1-P_0$ are algebraically or Von-Neumann equivalent as projectors (see appendix A for more on this). Another notion of equivalence of projectors is given by unitary equivalence. In general Von Neumann equivalent projectors will not be unitarily equivalent. If however von Neumann equivalent projectors are embedded in higher dimensional matrices they can be made unitarily equivalent. In the 2×2 case for the shift operator example given above we embed the projectors I and $I - P_0$ as

$$I \to \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \qquad I - P_0 \to \begin{pmatrix} I - P_0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.3}$$

It is then straightforward to check that these matrices are related by the unitary transformation

$$Z = \begin{pmatrix} S & P_0 \\ 0 & S^\dagger \end{pmatrix} \tag{3.4}$$

with inverse

$$Z^{-1} = \begin{pmatrix} S^\dagger & 0 \\ P_0 & S \end{pmatrix}. \tag{3.5}$$

Note that S and therefore Z are not trace class as follows simply from the fact that the sum of eigenvalues of $\sqrt{S^\dagger S}$ is infinite.

A third notion of projector equivalence, and the one that we shall primarily exploit in this paper, is homotopy equivalence. For our purposes homotopy equivalence means that there exists a one parameter family of projection operators which interpolate between the two given projectors. Homotopy equivalence of projectors implies unitary equivalence, but not vice versa. Unitary equivalence in an algebra $\mathcal{A}$ however implies homotopy equivalence when embedded in $\mathcal{M}_2(\mathcal{A})$. Suppose E and F are unitary equivalent projectors satisfying $E = ZFZ^{-1}$, then there is a family of operators interpolating between

$$\hat{E} = \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$\hat{F} = \begin{pmatrix} F & 0 \\ 0 & 0 \end{pmatrix}.$$

The interpolating matrix is

$$W_t = \begin{pmatrix} (c^2 + Zs^2) & cs(1 - Z) \\ cs(Z^{-1} - 1) & c^2 + Z^{-1}s^2 \end{pmatrix} \tag{3.6}$$

where $c = \cos(\pi t/2)$ and $s = \sin(\pi t/2)$. In particular W_t at $t = 0$ is the identity and at $t = 1$ is $W = (\, Z \quad 0 \quad 0 \quad Z^{-1}\,)$. Since Z is unitary it follows that W_t is also unitary and its inverse is given by

$$W_t^{-1} = \begin{pmatrix} (c^2 + Z^{-1}s^2) & cs(Z - 1) \\ cs(1 - Z^{-1}) & c^2 + Zs^2 \end{pmatrix}. \tag{3.7}$$

Note in our application E, F, Z are themselves 2×2 matrices with entries which are operators in the Hilbert space of a free oscillator. The expression for Z is given above in (3.4). Therefore, $\hat{E}$ and $\hat{F}$ are 4×4 matrices with entries which are operators in the Hilbert space. More explicitly the matrix is

$$W_t = \begin{pmatrix} c^2 + Ss^2 & P_0 s^2 & cs(1 - S)- & csP_0 \\ 0 & c^2 + s^2 S^\dagger & 0 & cs(1 - S^\dagger) \\ cs(S^\dagger - 1) & 0 & c^2 + s^2 S^\dagger & 0 \\ cs(P_0) & cs(S - 1) & s^2 P_0 & c^2 + s^2 P_0 \end{pmatrix}. \tag{3.8}$$

4. Non-commutative tachyons

We now use the formalism discussed in the previous section to construct new solutions to the equations of motion from known ones in various settings. Of particular interest will be the homotopy equivalence of projection operators which leads to new solutions which interpolate between simple embeddings of recently constructed solutions into higher rank gauge groups. In this section we will consider solutions of unstable brane worldvolumes in the presence of B-fields, which represent lower branes. In the next section we will consider solutions of three-brane worldvolume theory which represent geometrical worldvolume deformations associated with D-string charge (fluxons). We will find that some of the results in this section shed light on the topology of string configuration space.

4.1 $D25$ in bosonic string and $D9$ in type IIA

The discussion in these two cases is almost identical. We will for simplicity discuss the construction of $D23$ in $D25$. Substituting unstable $D7$ inside unstable $D9$ of Type IIA is a trivial generalization.

Following the notation of [13] the bosonic string action for the tachyon and gauge field with vanishing B field is given by

$$S = \frac{c}{g_s} \int d^{26}x \sqrt{g} \left\{ -\frac{1}{4} h(\phi - 1) F^{\mu\nu} F_{\mu\nu} + \cdots + \frac{1}{2} f(\phi - 1) \partial^\mu \phi \partial_\mu \phi + \cdots - V(\phi - 1) \right\}, \tag{4.1}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]\,. \tag{4.2}$$

In this notation the closed string vacuum corresponds to $\phi = 1$ at which point the potential has a local minimum and vanishes along with h, i.e. $V(0) = 0 = h(0)$. The open string vacuum corresponds to $\phi = 0$ where the potential has a local maximum and has been normalized as $V(-1) = 1$.

The prescription for generalizing the action (4.1) to a constant B field background has been given in [20]. Specifically all products in the action are deformed to star products (which then become matrix products in the Hilbert space representation discussed in section 1) with noncommutativity parameter given in terms of B and the closed string coupling g_s and closed string metric $g_{\mu\nu}$ are replaced by the corresponding open string quantities G_s and $G_{\mu\nu}$.[1] The end result is the action

$$S = \frac{2\pi\theta c}{G_s} \int d^{24}x\sqrt{G}\ \mathrm{Tr}\Big\{ -\frac{1}{4}h(\phi-1)(F^{\mu\nu}+\Phi^{\mu\nu})(F_{\mu\nu}+\Phi_{\mu\nu}) + \cdots + \\ + \frac{1}{2}f(\phi-1)D^\mu\phi D_\mu\phi + \cdots - V(\phi-1)\Big\} \tag{4.3}$$

with

$$F_{24,25} + \Phi_{24,25} = -iF_{z\bar{z}} + \frac{1}{\theta} = -\frac{1}{\theta}[C,\bar{C}]\,. \tag{4.4}$$

There is some freedom in writing this action given by the choice of $\Phi_{\mu\nu}$. Following [13] we have made the choice given in (4.4).

In the U(N) case, C becomes an $N \times N$ matrix whose entries are elements in the Hilbert space of a free oscillator. In this case the leading terms in the action can be rewritten as

$$S = \frac{2\pi\theta c}{G_s} \int d^{24}x\sqrt{G}\ \mathrm{Tr}\Big\{ -\frac{1}{4}h(\phi-1)F^{mn}F_{mn} - \frac{1}{4\theta^2}h(\phi-1)([C,\bar{C}])^2 + \cdots + \\ + \frac{1}{2}f(\phi-1)D^\mu\phi D_\mu\phi + \cdots - V(\phi-1)\Big\}\,, \tag{4.5}$$

where we have used (4.4). The indices m, n extend from 1 to 24 and the trace now includes summing over the Hilbert space indices as well as the colour indices.

The solution generating transformation acts in the U(1) case as :

$$\begin{aligned} &\phi \to U\phi\bar{U}\,, \\ &C \to UC\bar{U}\,, \quad A_\mu \to\ UA_\mu\bar{U}\,, \quad \mu = 0\ldots 23\,. \end{aligned} \tag{4.6}$$

Acting on the closed string vacuum

$$\phi = 1\,, \qquad C = \bar{a}\,, \qquad A_\mu = 0$$

[1] The relations between these parameters can be found in e.g. [13].

we obtain the solution

$$
\begin{aligned}
\phi &= SS^\dagger = (1 - P_0) \\
C &= S\bar{a}S^\dagger \\
A_m &= 0\,.
\end{aligned}
\tag{4.7}
$$

This solution was identified in [13] as the D23 brane as its tension can be shown to be that of the D23 brane.

4.2 Interpolating between static solutions : general remarks

In the following we will apply the formalism developed in section 3 to interpolate between simple embeddings of U(1) solutions into U(4). Specifically we embed U(1) solutions such as the closed and open string vacua into U(4) in a trivial manner so that the U(4) field equations are satisfied. We then conjugate these solutions by $W(t)$ constructed in section 2, thereby generating new t-dependent solutions. A natural question to consider is the interpretation of the parameter t. The most straightforward interpretation is that it simply parametrizes a family of static classical solutions. In this sense the solutions are like a family of sphalerons parametrized by t (for recent discussions of these in string theory see [21, 22]). They are not exactly sphalerons because they do not have finite energy. The infinities in the energy are however easy to understand and can be regulated by working on a non-commutative T^2 rather than a non-commutative R^2. The parameter t just labels a static solution which lies at an intermediate point between the two solutions we are interpolating between.

Another possible interpretation of t is as euclidean time. Specifically the solution generating transformation also allows us to construct solutions to the euclidean space-time equations of motion by mapping the interpolating parameter t to the euclidean time x_0. There is a moduli space corresponding to choices of the function $x_0(t)$ subject to some boundary conditions. In this case we also conjugate the time-covariant derivative,

$$
D_0 \to W(x_0)\partial_t W(x_0)^{-1}\,. \tag{4.8}
$$

The first description above is just a special case of this moduli space where x_0 is chosen to be independent of t. We will see that the action of these solutions is infinite. In some case this infinity can be expected on physical grounds. In other cases, it is conceivable that some deformation of these solutions can give finite action (although we do not have any concrete directions for the right deformations at this point).

JHEP03(2001)037

4.3 Case I

Consider the following solution to the equations of motion of the bosonic string :

$$\Phi^{(0)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad C = \bar{a} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{4.9}$$

After applying the conjugating transformation

$$U = (\quad S \quad 0 \quad 0 \quad 0 \qquad 0 \quad 0 \quad 0 \quad 0 \qquad 0 \quad 0 \quad 0 \quad 0 \qquad 0 \quad 0 \quad 0 \quad 0 \,) \tag{4.10}$$

the configurations we get are:

$$\Phi = \begin{pmatrix} 1-P_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad C = \begin{pmatrix} S\bar{a}S^\dagger & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.11}$$

The interpolating configurations are

$$\begin{aligned} \Phi(t) &= W_t \Phi^{(0)} W_t^{-1} \\ &= \begin{pmatrix} c^4 + c^2 s^2 (S+S^\dagger) + s^4(1-P_0) & 0 & -c^3 s(1-S) - cs^3(1-S)S & c^3 s P_0 + cs^3 S P_0 \\ 0 & 0 & 0 & 0 \\ c^3 s(S^\dagger - 1) + cs^3 S^\dagger (S^\dagger - 1) & 0 & -c^2 s^2 (S + S^\dagger - 2) & -c^2 s^2 P_0 \\ c^3 s P_0 + cs^3 P_0 S^\dagger & 0 & -c^2 s^2 P_0 & c^2 s^2 P_0 \end{pmatrix} \end{aligned} \tag{4.12}$$

and

$$\begin{aligned} C &= W_t C^{(0)} W_t^{-1} \\ &= \begin{pmatrix} c^4 \bar{a} + c^2 s^2 (S\bar{a} + \bar{a}S^\dagger) + s^4 S\bar{a}S^\dagger & 0 & -c^3 s\bar{a}(1-S) - cs^3(S\bar{a} - S\bar{a}S) & c^3 s\bar{a}P_0 + cs^3 S\bar{a}P_0 \\ 0 & 0 & 0 & 0 \\ c^3 s(S^\dagger - 1)\bar{a} + cs^3(S^\dagger \bar{a} S^\dagger - \bar{a}) & 0 & c^2 s^2 (S^\dagger - 1)\bar{a}(S-1) & c^2 s^2 (S^\dagger - 1)\bar{a}P_0 \\ c^3 s P_0 \bar{a} + cs^3 P_0 \bar{a} S^\dagger & 0 & c^2 s^2 P_0 \bar{a}(S-1) & c^2 s^2 P_0 \bar{a} P_0 \end{pmatrix} \end{aligned} \tag{4.13}$$

where (4.11) is reproduced at $t = 1$.

For the moment we interpret t as the parameter in a one parameter family of static solutions and now compute the energy of the interpolating configurations. The contribution to the energy from the gauge field is proportional to the quantity

$$\begin{aligned} [C, \bar{C}]^2(t) &= W(t)[C, \bar{C}]^2(0) W(t)^{-1} \\ &= W(t)\,\mathrm{diag}(1, 0, 0, 0) W(t)^{-1} \\ &= \Phi(t)\,. \end{aligned} \tag{4.14}$$

The coefficient of proportionality is $h(\Phi(t) - 1)$, multiplying the two yields

$$h(\Phi(t) - 1)[C, \bar{C}]^2(t) = h(-1)(-\Phi(t) + 1)\Phi(t) = 0\,. \tag{4.15}$$

In the last step we have used the fact that $\Phi(t)$ is a projector, and $h(0) = 0$. Alternatively we can arrive at the same result by noting that $h(\Phi(0)-1)[C,\bar{C}]^2(0)$ is zero since h is zero in the first block and the field strength is zero in the other blocks. This quantity at time t is obtained by conjugating with $W(t)$, so this remains zero.

Since the solution is static there is furthermore no contribution to the energy from time derivatives. The only non-zero contribution to the energy therefore comes from the potential and is

$$\begin{aligned} \int d^{23}x \mathrm{Tr} V(\Phi - 1) &= \int d^{23}x \int V(-1)\mathrm{Tr}(1-\Phi) \\ &= \int d^{23}x \left(3\mathrm{Tr}(1) - \frac{1}{2} sin^2\left(\frac{\pi t}{2}\right) cos(\pi t)\right) \end{aligned} \tag{4.16}$$

The first term can be interpreted as the energy of the three $D25$ branes. The second term vanishes at $t = 0$ corresponding to the closed string vacuum and at $t = 1$ gives the energy of the $D23$ brane. At intermediate values of t the extra contribution to the energy can be either positive or negative.

The parameter t can also be interpreted as euclidean time. In this case while the fields Φ, C indeed take the values at $t = 0$ given in equation (4.9) and end at the configuration (4.11) at $t = 1$ the time derivative of Φ is non-zero, and in particular non-zero at $t = 0$ and $t = 1$. Specifically

$$\partial_t \Phi = [(\partial_t W)W^{-1}, \Phi] \tag{4.17}$$

which gives

$$\begin{aligned} \partial_t\Phi(0) &= \begin{pmatrix} 0 & 0 & (1/2)(S-1) & (1/2)P_0 \\ 0 & 0 & 0 & 0 \\ (1/2)(S^\dagger - 1) & 0 & 0 & 0 \\ (P_0/2) & 0 & 0 & 0 \end{pmatrix} \\ \partial_t\Phi(1) &= \begin{pmatrix} 0 & 0 & (1/2)S(S-1) & (1-S)(P_0/2) \\ 0 & 0 & 0 & 0 \\ (1/2)S^\dagger(S^\dagger - 1) & 0 & 0 & 0 \\ (P_0/2)(1-S^\dagger) & 0 & 0 & 0 \end{pmatrix} \end{aligned} \tag{4.18}$$

The covariant derivative is of course zero.

It is possible to construct solutions where the time derivatives at the beginning and end points are zero. This is achieved by essentially performing a redefinition of the time variable such that the interval $[0,1]$ is streched to $[-\infty, \infty]$. For example we can let euclidean time x_0 be a function of the interpolating parameter t as $x_0 = \tan\pi(t - 1/2)$ The appropriate solution is obtained by conjugating with a matrix $\mathrm{U}(x_0) = W(t(x_0))$.

A natural quantity to consider for euclidean solutions is the value of the action. For the solution constructed above this is easy to compute using our previous computations for the energy of the one parameter family of static solutions. In particular the only non-zero contribution again comes from the potential V giving us

$$S = \frac{2\pi\theta c}{G_s} \int d^{23}x dx_0 \mathrm{Tr} V(\Phi(t(x_0)) - 1)\,, \tag{4.19}$$

where $\int d^{23}xV(\Phi(t) - 1)$ is given in (4.16). It follows that the action in this case will contain the usual divergence from the $d^{23}x$ integral as well as the Tr (1) divergence. Both could be regulated by working on compact commutative and noncommutative spaces respectively. The action however contains another divergent contribution coming from the x_0 integral. For the Tr (1) term this follows because it is x_0 independent while the integral of the remaining x_0 dependent term diverges at the $x_0 \to \infty$ limit.

The divergent x_0 integral can be partially cured by instead letting t run from 0 to 2 and stretching this interval to infinity by e.g. the map $x_0 = \tan(\pi/2)(t-1)$. The euclidean time integral of the x_0 dependent term in the potential, the second term in (4.16), now becomes finite. There is still an infinite $\int dx_0$ multiplying Tr (1). This solution corresponds to 'evolving' in x_0 from the vacuum configuration (4.9) through the solution (4.11) and then back to (4.9). It can further be modified to ensure that the time derivative ∂_0 is zero when the configuration passes through (4.11) at $t = 1$. There is a large class of such solutions interpolating from one vacuum to the same parametrized by a choice of finite function living on the interval $[0, 2]$. These solutions are in some sense like an instanton-anti-instanton pair. The endpoints of the solution are the mixed closed string vacuum, open string vacuum configurations given in (4.9). At some finite x_0 the configuration consists of an unstable $D23$ brane and the open string vacuum as in (4.11). Therefore at least for these configurations the open string vacua appearing in the solution are basically just spectators with the evolution all occuring in one particular $U(1)$ sector. This action also splits into a sum of terms associated with each block. The Tr (1) part corresponds to the three $D25$ branes in their vacuum states while the remaining finite contribution corresponds to the closed string vacuum - D23 brane - closed string vacuum sector. Of course at other values of x_0 the solution $\Phi(t(x_0))$ has off-diagonal elements so it is difficult to make this interpretation rigorous.

4.4 Some other interpolating solutions

If we start with a general diagonal configuration of fields which can be interpreted simply, we are not guaranteed to end with a diagonal configuration at $t = 1$ after conjugation by W. Two more examples where this is possible are given as follows. If we start with $\Phi = \mathrm{diag}(0, 1 - P_0, 1, 1)$, we also end up with something diagonal $\Phi = \mathrm{diag}(0, 1, 1, 1)$. Another pair of diagonal solutions we can interpolate between is $\Phi = \mathrm{diag}(1, 1, 1, 1), C = \bar{a}\,\mathrm{diag}(1, 0, 0, 0)$ and $\Phi = \mathrm{diag}(1, 1, 1, 1), C = S\bar{a}S^\dagger\,\mathrm{diag}(1, 0, 0, 0)$

4.5 $D - \bar{D}$ system, elementary strings and a puzzle

For the superstring, unlike the bosonic string, there are stable D-branes which carry conserved charges from the Ramond-Ramond sector (There are of course also D-branes which are uncharged and unstable as in the bosonic string as well, but our primary interest in this section will be with the stable ones.). If a stable D-brane and it's oppositely charged version, the $\bar{D}$-brane, are brought together, they can annihilate. This instability manifests itself from the worldsheet point of view through a complex tachyon field which is charged under the $\mathrm{U}(1) \times \mathrm{U}(1)$ gauge symmetry of the $D(\bar{D})$-brane respectively, i.e. the tachyon transforms in the bifundamental of the $\mathrm{U}(1) \times \mathrm{U}(1)$ gauge symmetry. In the noncommutative case ϕ transforms as

$$\phi \to U\phi\bar{V}\bar{\phi} \to V\bar{\phi}\bar{U}\,. \tag{4.20}$$

under a gauge transformation, and more generally under the solution generating transformation. Similarly the gauge fields in the noncommutative directions transform as

$$C^{+} \to UC^{+}U^{\dagger}C^{-} \to VC^{-}V^{\dagger}\,. \tag{4.21}$$

We now apply the previous discussion to the $D9 - \bar{D}9$ system. This leads to some rather unexpected phenomena when we embed in $\mathrm{U}(4)$ as before. In particular it appears that we can construct an interpolating solution between any two configurations consisting of stable $D7$ and stable $\bar{D}7$ branes . To see this take the following field configuration for the tachyon and gauge fields in the noncommutative directions

$$\begin{aligned}\Phi &= \mathrm{diag}(1,0,0,0)\\ C^{+} &= \mathrm{diag}(a^{\dagger},0,0,0)C^{-} = \mathrm{diag}(a^{\dagger},0,0,0)\,.\end{aligned} \tag{4.22}$$

Now conjugate this 'ground' state by

$$U = (W_t)^n\,, \qquad V = (W_t)^m\,. \tag{4.23}$$

The time dependent tachyon becomes $\phi(t) = (W_t)^n\phi(W_t^{\dagger})^m$, and similarly for the gauge fields. Evaluating at $t = 1$ we find that

$$\begin{aligned}\phi(1) &= \mathrm{diag}(S^n(S^{\dagger})^m,0,0,0)\\ C^{+}(1) &= \mathrm{diag}(S^n a^{\dagger}(S^{\dagger})^n,0,0,0)C^{-}(1) = \mathrm{diag}(S^m a^{\dagger}(S^{\dagger})^m,0,0,0)\,.\end{aligned} \tag{4.24}$$

The (1,1) component of these fields is exactly the $n - D7, m - \bar{D}7$ configuration identified by [13]. Furthermore it is easy to check that by applying $U = W_t^{\dagger}$ and $V = 1$ or $U = 1$ and $V = W_t^{\dagger}$ one can remove the number of $D7$ and $\bar{D}7$ branes respectively. This is not what we would have naively expected to be the case.

The same constructions can be used to construct closed string solitons in noncommutative gauge theory. Some relevant papers are [17, 23]. Here the solutions involve $\Phi = 1$ and $E = P_0$. Consider the U(4) configuration $\Phi = \mathrm{diag}(1,1,1,1), E = (P_0,1,0,0)$ The tachyon is sitting at the closed string minimum here in all four blocks, but the electric flux describes an elementary string soliton in the first block. The second block may be interpreted as describing infinitely many closed strings. Conjugating by $\mathrm{diag}(Z, Z^{-1})$ we get $\Phi = \mathrm{diag}(1,1,1,1), E = (0,1,0,0)$, i.e. we have got rid of the closed string in the first block. By embedding further in U(7) we can interpolate to a configuration where the elemenatry string has been replaced by an unstable brane of codimension 2. Starting with $\Phi = \mathrm{diag}(1;1,1,1;0,0,0)$, $E = (P_0;1,0,0;0,0,0)$ we can interpolate to $\Phi = \mathrm{diag}(1;1,1,1;0,0,0)$, $E = (0;1,0,0;0,0,0)$ using matrices which act non-trivially on the first and the second set of three entries. Then using matrices acting on the first entry and the second set of three entries we can get to $\Phi = \mathrm{diag}(1-P_0;1,1,1;0,0,0)$, $E = (P_0;1,0,0;0,0,0)$. In this way we have interpolated, in the first block, between an elementary string soliton and an unstable brane by passing through the vacuum.

That we could interpolate between vacuum configurations and these configurations which contain charged objects like a BPS D-brane or a closed elementary string seems somewhat surprising. These interpolating solutions are not finite action instanton solutions so there is no immediate contradiction involving transitions from a direct sum of open and closed string vacua to a state containing a charged brane, or transitions from the vacuum to charged brane and back to the vacuum. However even the special case of $\partial_t x_0 = 0$, where we do not view these as solutions to euclidean equations of motion but rather as families of on-shell configurations, is intriguing. It shows that there is a family of static solutions which connect the vacuum to the brane configuration. In the case where the brane being created is a D23-brane or an unstable D7-brane as in the earlier subsection 4.2, the existence of such configurations is expected from the discussion of the topology of string configuration space where unstable branes are interpreted as sphalerons [21]. The fact that the family of configurations is on-shell is not predicted by that discussion, but its existence is not surprising. However the interpolation in the case of charged branes implies that the conservation of RR charge is not related to the topology of string configuration space but to some more subtle topology whose relation to the topology of string configuration space remains to be clarified. The existence of these interpolations can again to some extent be anticipated by the considerations of [24]. There it was shown, by CFT arguments, that while magnetic flux on T^2 labels different sectors in the space of fields for Yang-Mills on the torus, configurations with different flux can be connected in string configuration space. In that case one needed the stringy description to see the interpolation but here the fields of NC Yang-Mills suffice.

Another possible solution of the puzzle which we consider less likely, but are unable to dismiss completely, is that the naive interpretation of the block diagonal configuration, which leads to the conclusion that the charge of the block diagonal configuration is just the sum of charges associated with each block, is missing some subtlety of the full non-abelian brane world-volume action, and actually has zero brane charge. One might be tempted to think this is the case because the solutions can be generated in the U(4) case by conjugating the vacuum with an operator of index 0. Connections between the index of the conjugating operator and charge have been discussed in [19, 25, 26] for example but it is hard to see, in terms of brane actions how the direct sum configuration could fail to have the direct sum of charges.

Since we have seen that this solution generating technique has implications for the string configuration space and leads to an interesting puzzle, it is instructive to study its implications in a context where there are no tachyons and the system is as simple is as possible. In the following we look at the fluxons and show that sensible interpolations between U(4) configurations which have a clear meaning in terms of $D3$ and $D1$ branes is possible.

5. Fluxons and noncommutative gauge theory

We review here a class of solutions of $\mathcal{N} = 4$ supersymmetric noncommutative $U(1)$ gauge theory discussed in [19, 27, 14, 15]. The action (with x^1 and x^2 the noncommutative directions) is given by

$$S = \frac{2\pi\theta}{g^2}\int dtdx^3\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\mu\Phi_a D^\mu\Phi_a - \frac{1}{4}[\Phi_a,\Phi_b]^2\right] + \text{fermions} \tag{5.1}$$

where the covariant derivative is given by

$$D_\mu = \partial_\mu - iA_\mu \tag{5.2}$$

and the curvature tensor by

$$F_{\mu\nu} = i([D_\mu, D_\nu] - i\theta_{\mu\nu})\,. \tag{5.3}$$

The equations of motion following from this action are

$$[D^\nu,[D_\nu,D_\mu]] = [\Phi_a\,,[D_\mu,\Phi_a]][D^\mu,[D_\mu,\Phi_a]] = [\Phi_b,[\Phi_b,\Phi_a]]\,. \tag{5.4}$$

Now consider static solutions with no electric potential $A_0 = 0$ and only one nontrivial scalar field Φ. Defining the magnetic field as

$$B_i := \frac{1}{2}\epsilon_{ijk}F_{jk} \tag{5.5}$$

one can show that the first order BPS equations

$$B_i + [D_i,\Phi] = 0 \tag{5.6}$$

are consistent with the equations of motion. Working in the gauge $A_3 = 0$ the BPS equations reduce to

$$\frac{\partial \Phi}{\partial x^3} = \frac{1}{\theta}([C, \bar{C}] + 1)$$
$$\frac{\partial C}{\partial x^3} = -[C, \Phi] \frac{\partial \bar{C}}{\partial x^3} = [\bar{C}, \Phi] \tag{5.7}$$

where we have defined C and $\bar{C}$ as

$$C := (a^\dagger + i\sqrt{\theta} A)\,, \qquad \bar{C} = (a - i\sqrt{\theta}\bar{A}) \tag{5.8}$$

with A and $\bar{A}$ given in terms of A_1 and A_2 as

$$A := \frac{A_1 - iA_2}{\sqrt{2}}\,, \qquad \bar{A} := \frac{A_1 + iA_2}{\sqrt{2}}\,. \tag{5.9}$$

C and $\bar{C}$ are just covariant z and $\bar{z}$ derivatives respectively rescaled by $\sqrt{\theta}$.

Before looking for solutions to these equations it is interesting to note that there are different symmetries of the equations of motion. The most obvious from (5.4) is the conjugation transformation

$$D_\mu \to S D_\mu S^\dagger, \Phi_a \to S \Phi_a S^\dagger \tag{5.10}$$

that we have been discussing so far. In this setting we call this symmetry the "non-BPS symmetry" because in general a solution to the BPS equations will not map back to a solution of the BPS equations under this symmetry, but rather only to a solution of the full equations of motion. The BPS equations however have another symmetry [14, 15] which takes one BPS solution to another BPS solution. This follows by defining the field

$$\Phi^{(P)} := \Phi - \frac{x^3}{\theta}\,. \tag{5.11}$$

The first BPS equation then becomes

$$\frac{\partial \Phi^{(P)}}{\partial x^3} = \frac{1}{\theta}[C, \bar{C}] \tag{5.12}$$

while the remaining BPS equations take the same form with Φ replaced by $\Phi^{(P)}$. The BPS symmetry is then given by conjugating all the fields as above, i.e.,

$$\Phi^{(P)} \to S \Phi^{(P)} S^\dagger, C \to S C S^\dagger \tag{5.13}$$

and similarly for $\bar{C}$.

We now discuss a few simple solutions to the BPS equations and the solutions generated from them under the BPS and non-BPS symmetry transformations respectively. The simplest solution is the ground state given by

$$\Phi = \Phi_0, C = a^\dagger, \bar{C} = a, B_3 = 0\,. \tag{5.14}$$

Acting on the ground state having $\Phi_0 = 0$ with the BPS symmetry transformation one finds the fluxon solution [19, 14, 15]

$$\Phi = \frac{x^3}{\theta} P_0 \,, \qquad C = S a^\dagger S^\dagger \,, \qquad \bar{C} = S a S^\dagger \,, \qquad B_3 = -\frac{1}{\theta} P_0 \,. \tag{5.15}$$

This solution corresponds to a D-string piercing a D3-brane located in the x^1, x^2, x^3 plane. The D-string is associated with a geometrical deformation of the D3-brane worldvolume. The deformation is localized near the origin of the (x_1, x_2) plane, and takes the form of a spike extending in the (Φ, x_3) plane at an angle $1/\theta$ (see for example [14]). Further evidence for this picture is given by considering the charges of this configuration. The term $\int C_{03} \wedge F_{12}$ shows that this configuration has $D1$ charge in the 3 direction. The term $\int C_{04} \wedge F_{12} \wedge [\partial_3, X^4]$ coming from the non-abelian pull-back [28] gives D1-charge along the $\Phi \equiv X^4$ direction. It is noteworthy that the latter charge has an extra factor of $\frac{1}{\theta}$ consistent, for large θ (where $\tan(1/\theta) \sim \frac{1}{\theta}$), with the geometrical picture.

Acting on (5.14) with the non-BPS symmetry transformation one generates the solution

$$\Phi = \Phi_0 (1 - P_0),\, C = S a^\dagger S^\dagger \,, \qquad \bar{C} = S a S^\dagger \,, \qquad B_3 = \frac{1}{\theta} P_0 \,. \tag{5.16}$$

This solution corresponds to a D3-brane located at $\Phi = \Phi_0$ along most of the (x^1, x^2) plane and having a deformation localized near $x^1 = 0 = x^2$ and extending to $\Phi = 0$.

There are also interesting solutions with $C = 0 = \bar{C}$ is given by

$$\Phi = \Phi_0 + \frac{x^3}{\theta} \,, \qquad C = 0 = \bar{C} \,, \qquad B_3 = -\frac{1}{\theta} \,. \tag{5.17}$$

This configuration carries infinite D-string charge along x_3 and along Φ through Chern-Simons couplings discussed above. This can be confirmed by a fluctuation analysis which shows that momentum modes along the (x_1, x_2) directions have zero energy. We may also understand this picture by noting that if one conjugates the ground state solution (5.14) by S^n under the BPS symmetry then Φ becomes in that case $\Phi = (x^3/\theta) P_n$ where P_n projects onto an n-dimensional subspace. This solution is the n fluxon solution corresponding to n D-strings piercing the D3-brane. As n is taken to infinity one obtains (5.17) (with $\Phi_0 = 0$).

Conjugating (5.17) by the BPS symmetry one finds

$$\Phi = \left(\Phi_0 + \frac{x^3}{\theta} \right) - \Phi_0 P_0 \,, \qquad C = 0 = \bar{C} \,, \qquad B_3 = -\frac{1}{\theta} \tag{5.18}$$

while the non-BPS symmetry yields

$$\Phi = \left(\Phi_0 + \frac{x^3}{\theta} \right) (1 - P_0) \,, \qquad C = 0 = \bar{C} \,, \qquad B_3 = -\frac{1}{\theta} \,. \tag{5.19}$$

The BPS solution represents a D3-brane with infinite D-string charge inclined in the (x_3, Φ) plane and shifted by Φ_0 everywhere in the x_1, x_2 plane except near the origin. The non-BPS solution represents a D3-brane with infinite D-string charge inclined in the x_3, Φ plane and having a D-string near the origin of the x_1, x_2 plane intersecting it and pointing along $\Phi = 0$ in the (Φ, x_3) plane.

6. Interpolating from vacuum to fluxon

Interpolating solutions can be constructed after embedding in U(4). Consider a U(4) solution obtained by a diagonal embedding of one copy of (5.16) and three copies of the solution in (5.17) with $\Phi_0 = 0$.

$$\begin{aligned} \Phi &= \frac{x^3}{\theta}\,\mathrm{diag}(P_0, 1, 1, 1) \\ \Phi^{(P)} &= \frac{-x^3}{\theta}\,\mathrm{diag}(1 - P_0, 0, 0, 0) \\ B_3 &= \frac{-1}{\theta}\,\mathrm{diag}(P_0, 1, 1, 1)\,. \end{aligned} \tag{6.1}$$

The first entry along the diagonal describes a fluxon. The remaining entries describe sheets carrying infinite D-string charge at an angle in Φ, x_3 plane.

This is related to the following configuration by BPS symmetry :

$$\begin{aligned} \Phi &= \frac{x^3}{\theta}\,\mathrm{diag}(0, 1, 1, 1) \\ \Phi^{(P)} &= -\frac{x^3}{\theta}\,\mathrm{diag}(1, 0, 0, 0) \\ B_3 &= -\frac{1}{\theta}\,\mathrm{diag}(0, 1, 1, 1) \end{aligned} \tag{6.2}$$

Multiplying (6.2) by $W(t)$ of (3.6) on the left and $W^{-1}(t)$ on the right yields a family of solutions starting at $t = 0$ from (6.2) and ending at $t = 1$ at (6.1). The interpolation removes the D-string fluxon from the first three-brane, in the presence of three extra three-branes carrying infinite D-string charge.

One picture of what is happening in the interpolation between (6.1) and (6.2) is that the D-string breaks at the location at which it intersects the D3-brane while the endpoints remain attached to the D3-brane forming a monopole/anti- monopole pair. Pulling the monopole(anti-monopole) to $x^3 = \infty(-\infty)$ respectively we arrive at the solution described above. It would be interesting to test this picture by explicit investigations of the interpolating configurations.

Interpolating families can be constructed based on the non-BPS conjugation as well. They connect, for example,

$$\Phi = \frac{x^3}{\theta}\,\mathrm{diag}(1 - P_0, 0, 0, 0)$$

$$
\begin{aligned}
B_3 &= \frac{1}{\theta}\,\mathrm{diag}(1,1,1,1) \\
C &= \bar{C} = 0
\end{aligned}
\tag{6.3}
$$

to the solution

$$
\begin{aligned}
\Phi &= \frac{x^3}{\theta}\,\mathrm{diag}(1,0,0,0) \\
B_3 &= \frac{1}{\theta}\,\mathrm{diag}(1,1,1,1) \\
C &= \bar{C} = 0
\end{aligned}
\tag{6.4}
$$

Both initial and final configurations contain three 3-branes at $\Phi = 0$ carrying infinite D-string charge. The initial configuration has in addition a tilted D3 with infinite D-string charge, with a D-string piercing it. The final configuration has the piercing D-string removed.

7. Summary and outlook

We have looked at families of static on-shell open string configurations which interpolate between different classical solutions as well as time-dependent solutions of the euclidean equations of motion. Our interpolating technique comes from a basic theorem in the K-theory of operator algebras. It would be interesting to find the K-theoretic significance (along the lines of [10]–[12], [26, 29]) of the fact that we can interpolate between these non-commutative solutions when the rank of the gauge group becomes precisely such that the different definitions of projector eqivalence become identical.

We have so far not mentioned time-dependent solutions for Minkowski signature. In fact such solutions can be generated in exactly the same manner as the euclidean solutions. For example it is easy to generalize the interpolating solution in (4.12) and (4.13) to the case where t is the time coordinate in Minkowski space. It turns out that if we perform a naive regulation of the trace by cutting off the states in the Hilbert space at finite level number to make the energy well-defined, then this energy is not conserved. This is trivial to see in that we start with the mixed closed/open string vacuum configuration in (4.9) and end in the D23 brane, open string vacua configuration in (4.11). The existence of formal solutions violating energy conservation can actually be seen in the ordinary wave equation. However in that case these solutions are in general unphysical and discarded. In the case at hand though we see no obvious reason why such solutions should be discarded. We hope to explore the minkowskian solutions further in the near future.

We do not expect the solutions discussed in 4.5 to be deformable to any finite action ones but it is conceivable that those in section 4.3 can be so deformed. It would be very interesting to exhibit finite action solutions which accomplish the kind of vacuum to brane or vacuum to vacuum via a brane that we described. Repacing the NC R^2 by a NC T^2 would cure some of the infinities we had but it seems unlikely to cure all of them.

We made some comments on the relation of these families of non-commutative solitons to the works of [21, 24, 22] in section 4. We expect that further exploration of these families of NC solitons will have interesting implications for the topology of the configuration space of string fields. While the existence of the unstable D-branes can be understood by interpreting them as sphalerons associated with certain instantons in string theory, it is natural to ask, for example, if there is a similar interpretation for the existence of the families of interpolating solutions constructed here.

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A. Some K-theory concepts

We review some facts about K-theory of operator algebras, highlighting some concepts and formulae, while referring to standard sources like [31] for a detailed exposition of definitions and proofs. These facts are used in the construction of solutions in the text.

Let A be an algebra of operators in a Hilbert space. $K_0(A)$, the K-group of an algebra A is defined in terms of equivalence classes of projectors. A projector in A is an element $p \in A$ which satisfies $p^2 = p$. It is the space of inequivalent projectors in $M_\infty(A)$, where $M_\infty(A)$ is the algebra of large N matrices with entries taking values in A. There are different definitions of equivalence. Two projectors p and q are said to be algebraically equivalent if there are elements $x, y \in A$ such that $xy = P, yx = Q, x = Px = xQ = PxQ, y = Qy = yP = QyP$. Two projectors P and Q are said to be equivalent by similarity if they conjugates by a unitary element z, i.e $zPz^{-1} = Q$. Two projectors are said to be homotopically equivalent if there is a family of projectors connecting P and Q.

While these definitions are not equivalent in A they become equivalent when we consider matrices whose entries take values in A. The first relevant result is that if P and Q are algebraically equivalent, then

$$\begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix} = Z \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix} Z^{-1}, \tag{A.1}$$

where $Z = \begin{pmatrix} y & 1-Q \\ 1-P & x \end{pmatrix}$ and $Z^{-1} = \begin{pmatrix} x & 1-e \\ 1-f & y \end{pmatrix}$. The second result is that if Z is invertible then there is a path of invertibles in $M_2(A)$ from 1 to $\mathrm{diag}(Z, Z^{-1})$. The interpolating matrix $W(t)$ is $\mathrm{diag}(x,1) \cdot u(t) \cdot \mathrm{diag}(y,1) \cdot u(t)^{-1}$ where

$$u(t) = \begin{pmatrix} \cos(\pi t/2) & -\sin(\pi t/2) \\ \sin(\pi t/2) & \cos(\pi t/2) \end{pmatrix} .$$

These basic K-theoretic constructions are used in section 3, where, in the simplest cases, the role of P, Q is played by $1, 1 - P_0$, and the role of x, y is played by $S, S^\dagger$.

References

[1] N. Nekrasov and A. Schwarz, *Instantons on noncommutative* $\mathbb{R}^4$ *and* $(2,0)$ *superconformal six dimensional theory*, *Comm. Math. Phys.* **198** (1998) 689 [hep-th/9802068].

[2] P.-M. Ho, *Twisted bundle on noncommutative space and* U(1) *instanton*, hep-th/0003012.

[3] R.Gopakumar, S.Minwalla, and A.Strominger, *Noncommutative solitons*, *J. High Energy Phys.* **05** (2000) 020 [hep-th/0003160].

[4] R. Gopakumar, S. Minwalla and A. Strominger, *Symmetry restoration and tachyon condensation in open string theory*, hep-th/0007226.

[5] M. Aganagic, R. Gopakumar, S. Minwalla and A. Strominger, *Unstable solitons in noncommutative gauge theory*, hep-th/0009142.

[6] D. Bak, *Exact multi-vortex solutions in noncommutative abelian-Higgs theory*, *Phys. Lett.* **B 495** (2000) 251 [hep-th/0008204].

[7] A.P. Polychronakos, *Flux tube solutions in noncommutative gauge theories*, *Phys. Lett.* **B 495** (2000) 407 [hep-th/0007043].

[8] D. Bak, S. U., K. Lee and J.-H. Park, *Noncommutative vortex solitons*, hep-th/0011099.

[9] A. Hashimoto and N. Itzhaki, *Traveling faster than the speed of light in noncommutative geometry*, hep-th/0012093.

[10] E. Witten, *D-branes and K theory*, *J. High Energy Phys.* **12** (1998) 019 [hep-th/9810188].

[11] E. Witten, *Overview of k-theory applied to strings*, hep-th/0007175.

[12] R. Minasian and G. Moore, *K-theory and Ramond-Ramond charge*, *J. High Energy Phys.* **11** (1997) 002 [hep-th/9710230].

[13] J.A. Harvey, P. Kraus and F. Larsen, *Exact noncommutative solitons*, *J. High Energy Phys.* **12** (2000) 024 [hep-th/0010060].

[14] K. Hashimoto, *Fluxons and exact BPS solitons in non-commutative gauge theory*, *J. High Energy Phys.* **12** (2000) 023 [hep-th/0010251].

[15] M. Hamanaka and S. Terashima, *On exact noncommutative BPS solitons*, *J. High Energy Phys.* **03** (2001) 034 [hep-th/0010221].

[16] S.P. de Alwis and A.T. Flournoy, *Some issues in noncommutative solitons as D-branes*, hep-th/0011223.

[17] J.A. Harvey, P. Kraus, F. Larsen and E.J. Martinec, *D-branes and strings as non-commutative solitons*, *J. High Energy Phys.* **07** (2000) 042 [hep-th/0005031].

[18] K. Dasgupta, S. Mukhi and G. Rajesh, *Noncommutative tachyons*, *J. High Energy Phys.* **06** (2000) 022 [hep-th/0005006].

[19] D.J. Gross and N.A. Nekrasov, *Monopoles and strings in noncommutative gauge theory*, *J. High Energy Phys.* **07** (2000) 034 [hep-th/0005204].

[20] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *J. High Energy Phys.* **09** (1999) 032 [hep-th/9908142].

[21] J.A. Harvey, P. Hořava and P. Kraus, *D-sphalerons and the topology of string configuration space*, *J. High Energy Phys.* **03** (2000) 021 [hep-th/0001143].

[22] N. Drukker, D.J. Gross and N. Itzhaki, *Sphalerons, merons and unstable branes in AdS*, *Phys. Rev.* **D 62** (2000) 086007 [hep-th/0004131].

[23] A. Sen, *Fundamental strings in open string theory at the tachyonic vacuum*, hep-th/0010240.

[24] A. Sen, *Uniqueness of tachyonic solitons*, *J. High Energy Phys.* **12** (2000) 001 [hep-th/0009090].

[25] K. Furuuchi, *Topological charge of* U(1) *instantons on noncommutative* $\mathbb{R}^4$, hep-th/0010006.

[26] J.A. Harvey and G. Moore, *Noncommutative tachyons and k-theory*, hep-th/0009030.

[27] D.J. Gross and N.A. Nekrasov, *Solitons in noncommutative gauge theory*, hep-th/0010090.

[28] R.C. Myers, *Dielectric-branes*, *J. High Energy Phys.* **12** (1999) 022 [hep-th/9910053].

[29] V. Mathai and I.M. Singer, *Twisted k-homology theory, twisted Ext-theory*, hep-th/0012046.

[30] I. Singer, *Lectures on K-theory for physicists*, MIT, Fall 2000.

[31] B. Blackadar, *K-theory of operator algebras*, Springer-Verlag, Berlin 1986.